# High Frequency Thin-Airfoil Theory for Subsonic Flow

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**A large parameter solution procedure of Schwartzschild and Landahl is adapted to the task of calculating closed-form approximate solutions for the pressure and lift of a flat-plate, infinite-span airfoil. Two general cases are treated: 1) the two-dimensional subsonic flow problem, in which the large parameter is the upwash frequency, and 2) the three-dimensional incompressible flow problem, in which the large parameter is the spanwise wavenumber of the upwash. For the first case, the four problems of a gust drifting with the freestream, a gust moving at other than the freestream velocity, a plunging motion, and a linear upwash are treated. For the second case, the two problems of a gust drifting with the freestream and a generalized gust moving at other than the freestream velocity are considered. Comparison of the solutions with available numerical results generally shows good agreement when the appropriate parameter is large. The solutions for the gust convecting with the freestream for both the two-dimensional compressible and the three-dimensional incompressible cases were derived previously by Adamczyk using the Wiener-Hopf technique.**

## Nomenclature

| | |
|---|---|
| $b$ | = semichord |
| $E,E^*$ | = combination of Fresnel integrals; Eq. (15); * denotes complex conjugate |
| erf | = error function |
| $H_n^{(2)}$ | = Hankel function |
| $k,k_x$ | = reduced frequency, $\omega b/U$ |
| $k^*$ | $= k/\beta^2$ |
| $k_y$ | = spanwise wavenumber normalized by $b$ |
| $L$ | = lift normalized by $(2\pi b\rho_0 U w_0/\beta)\times e^{i\omega t}$ |
| $M$ | = Mach number |
| $\mathfrak{M}$ | = quarter-chord moment normalized by $4b^2\pi\rho_0 U w_0 e^{i\omega t}$; positive counterclockwise |
| $P$ | = perturbation pressure on upper airfoil surface |
| $p$ | = leading or trailing-edge contribution to perturbation pressure |
| $r$ | $= 2-x$ |
| $T$ | = transformed time; Eq. (3) |
| $U$ | = freestream velocity |
| $w$ | = normal velocity on airfoil surface |
| $x,y,z$ | = chordwise, spanwise, and normal Cartesian coordinates normalized by $b$ |
| $X,Z$ | = transformed coordinates; Eq. (3) |
| $\beta$ | $= (1-M^2)^{1/2}$ |
| $\lambda$ | = defines generalized gust; Eq. (21) |
| $\mu$ | $= Mk^*$ |
| $\rho_0$ | = freestream density |
| $\Phi$ | = velocity potential |
| $\phi$ | = velocity potential with time removed |
| $\psi$ | = contribution to velocity potential; e.g., Eq. (9) |
| $\omega$ | = circular frequency |
| $D/Dt$ | = substantial derivative |

*Subscripts*

| | |
|---|---|
| $g$ | = generalized gust |
| $s$ | = Sears gust |
| $0$ | = plunging upwash |
| $1$ | = linear upwash |

*Superscripts*

| | |
|---|---|
| $0,1,2$ | = zero-, first- and second-order solutions |
| $1\ell,1t$ | = first-order leading- and trailing-edge contributions |

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## Introduction

THE two-dimensional linearized problem of an infinite-span airfoil with an unsteady upwash in a compressible flow has received wide attention. There have been several calculations (e.g., Refs. 1-4) that lead to numerical results for an airfoil undergoing unsteady motions or encountering a gust. However, an approximate closed-form solution is often preferable to tabular results. For low frequencies, such a solution is available already.[5,6] The present paper presents a high frequency solution procedure, which is based on work by Schwartzschild[7] and Landahl,[8] and which generalizes the work of Adamczyk.[9]

The basic idea behind the solution technique is that at high frequency (large chord/acoustic wavelength) it becomes possible to separate leading- and trailing-edge effects from one another. This then allows a leading-edge solution to be calculated as if the trailing edge were absent (i.e., making the chord semi-infinite). Similarly, a trailing-edge correction to the leading-edge solution can be calculated, and the process can be continued, obtaining an infinite convergent series of solutions. By applying the procedure properly, it is possible to derive closed-form approximate solutions. By comparison with the tabulated results available in literature, the accuracy of the approximate solutions can be evaluated.

The previous solution procedure of Landahl and Schwartzschild also is applied to the case of a skewed gust in incompressible flow. For this case, the large parameter is the spanwise wavenumber of the gust. Whereas the solutions for the two-dimensional compressible are wavelike in character and expressed in terms of Fresnel integrals, the solutions for the three-dimensional incompressible problems are hydrodynamic in character, and are expressed in terms of error functions.

## Problem Formulation for Two-Dimensional Compressible Flow

A flat-plate airfoil lies in the $z=0$ plane between $0\le x\le 2$ in a flow of Mach number $M$. (The spatial corrdinates are normalized by the semichord $b$.) The linearized equation for the velocity potential $\Phi$ is

$$\left[\nabla^2 - M^2\left(\frac{b}{U}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2\right]\Phi(x,z,t)=0 \qquad (1)$$

with the boundary conditions (BC's)

$$\Phi(x,0,t)=0 \qquad x\le 0 \qquad (2a)$$

$$\Phi_z(x,0,t) = bw(x)e^{i\omega t} \qquad 0<x\le 2 \tag{2b}$$

$$D\Phi/Dt=0 \qquad x\ge 2 \tag{2c}$$

where $D/Dt \equiv \partial/\partial t + (U/b)\partial/\partial x$ is the substantial derivative. A sinusoidal time dependence is assumed, so that $\Phi(x,z,t) = \phi(x,z)\exp(i\omega t)$. The reduced frequency is defined as $k \equiv \omega b/U$. Also, the abbreviations $\beta^2 \equiv 1-M^2$, $k^* \equiv k/\beta^2$ and $\mu \equiv Mk^*$ will be useful.

By transforming the coordinates and time according to

$$x \to X \qquad z \to Z/\beta \qquad \omega t \to T - M\mu X \tag{3}$$

the following boundary value problem for the potential $\phi$ results

$$\phi^*_{XX} + \phi^*_{ZZ} + \mu^2\phi^* = 0 \tag{4}$$

$$\phi^*(X,0) = 0 \qquad X \le 0 \tag{5a}$$

$$\phi^*_Z(X,0) = (b/\beta)w(X)e^{-i\mu MX} \qquad 0<X\le 2 \tag{5b}$$

$$(ik^* + \partial/\partial X)\phi^*(X,0) = 0 \qquad X \ge 2 \tag{5c}$$

where $\phi^*(X,Z) = \phi(x,z)$.

## Schwartzschild Solution

The solution of Schwartzschild will be used in solving the problem posed by Eqs. (4) and (5). Schwartzschild's basic solution can be started as follows: If a function $\phi^*$ satisfies the wave equation [Eq. (4)], together with the BC's

$$\phi^*(X,0) = F(X) \qquad X>0 \tag{6}$$

$$\phi^*_Z(X,0) = 0 \qquad X<0$$

then the solution can be written[8]

$$\phi^*(X,Z) = \frac{1}{\pi}\int_0^\infty G(X,\xi,Z)F(\xi)d\xi \tag{7a}$$

where

$$G(X,\xi,0) = (-X/\xi)^{1/2}[1/(\xi - X)]e^{-i\mu(\xi - X)} \qquad X<0 \tag{7b}$$

## Solution Procedure for Sears-Type Compressible Gust

Whereas the airfoil problem requires the application of three BC's given by Eq. (5), the Schwartzschild solution only can satisfy the two BC's given by Eq. (6). For this reason, the Schwartzschild solution will be applied in an iterative manner to find a series of solutions, all of which satisfy the condition of no flow through the airfoil [Eq. (5b)], but which satisfy only one of the two remaining BC's. By alternately correcting the upstream and downstream BC's, using Eq. (7), a convergent series of corrections will be obtained.[8]

Before Eq. (7) can be applied, a solution must be found which satisfies the condition of no flow through the airfoil surface without regard to the BC's upstream and downstream of the airfoil. Such a solution can be found, from a superposition of two-dimensional sources, to be

$$\phi^{(0)}(x,z) = \frac{ib}{2\beta}\int_{-\infty}^{\infty} e^{i\mu M(x-\xi)}$$

$$H_0^{(2)}\{\mu[(x-\xi)^2 + \beta^2 z^2]^{1/2}\}w(\xi)d\xi \tag{8}$$

This is the zeroth-order potential, expressed in real $x,z,t$ coordinates rather than the transformed $X,Z,T$ coordinates. Since $w(x)$ has not been specified for points not on the airfoil, the value used there is arbitrary. A judicious choice of $w(x)$ for $x$ not on the airfoil often will allow the integration in Eq. (8) to be performed, resulting in a simple closed form for $\phi^{(0)}$.

The zeroth-order potential $\phi^{*(0)}$ satisfies Eq. (4) and the BC for Eq. (5b), but does not satisfy either of the BC's given by Eqs. (5a) and (5c). The Schwartzschild solution now can be used to cancel $\phi^{*(0)}$ for $x<0$, although it does not affect the BC of no flow through the airfoil. The airfoil lies between $0 \le x \le 2$, but for the purpose of calculating the leading-edge correction $\psi^{(1)}$, it is assumed to extend downstream to infinity. Thus, $\psi^{(1)}$ corrects the upstream BC, but it does not correct the downstream BC. In the application of Eq. (7) the sign of $X$ must be changed, since the BC's for the airfoil problem have $\phi^*$ specified for $X<0$ and $\phi^*_Z$ for $X>0$, rather than vice versa, as given by Eq. (6). Application of Eq. (7), and inverse transforming to real $x,z,t$ space, gives for the leading-edge correction[8]

$$\psi^{(1)}(x,0) = -\frac{1}{\pi}\int_0^\infty (x/\xi)^{1/2} e^{-i\mu(1-M)(x+\xi)}$$

$$\phi^{(0)}(-\xi,0)\frac{d\xi}{x+\xi} \tag{9}$$

with $\phi^{(1)} = \phi^{(0)} + \psi^{(1)}$.

For the region downstream of the airfoil, the BC is $D\Phi/Dt=0$. Thus a function must be found to cancel $D\Phi^{(1)}/Dt$ for $x>2$. Since the pressure is related to the velocity potential by

$$Pe^{i\omega t} = -\rho_0(D\Phi/Dt) \tag{10}$$

it is more convenient to work directly with the pressure. Once $P$ is known, $\phi$ can be calculated by integration of Eq. (10); i.e.,

$$\phi(x,z) = -\frac{b}{\rho_0 U}\int_{-\infty}^{x} P(\xi,z)e^{-ik(x-\xi)}d\xi \tag{11}$$

Because $p$ and $\psi$ are linearly related, $p$ satisfies the wave equation. Also, requiring $p_z=0$ on the airfoil insures that $\psi_z=0$ there as well. Thus, the requirements for the application of Eq. (7) are met and the trailing-edge pressure correction in real $x,z,t$ coordinates is found to be

$$p^{(2)}(x,0) = -\frac{1}{\pi}\int_0^\infty (r/\xi)^{1/2} e^{-i\mu(1+M)(r+\xi)} P^{(1)}(2+\xi,0)\frac{d\xi}{r+\xi} \tag{12}$$

where $r=2-x$. The lower case $p$ is used here to denote the leading- or trailing-edge correction (corresponding to $\psi$ for the potential, (whereas $P^{(n)}$ represents the $n$/th-order solution for pressure (corresponding to $\phi^{(n)}$). Thus, $P^{(1)}$ and $\phi^{(1)}$ are related by Eq. (10) and $P^{(2)} = P^{(1)} + p^{(2)}$. Landahl [Ref. 8, Eq. (2.27)] gave a result identical to Eq. (12), except for the factor $1+M$, which was omitted, apparently by accident.

If $\phi^{(2)}$ is now calculated it will be found to be nonzero ahead of the airfoil. Thus, Eq. (9), with $\phi^{(0)}$ replaced by $\phi^{(2)}$, can be used to find a correction $\phi^{(3)}$, and the iteration procedure can be continued until sufficient accuracy is obtained.

If the Sears gust upwash is assumed at this point, a closed-form solution for $P^{(1)}$ can be found and a good approximation to $P^{(2)}$ made. As was mentioned, $w(x)$ in Eq. (8) is arbitrary for points off the airfoil. However, choosing

$$w_s(x) = -w_0 e^{-ikx} \tag{13}$$

for all $x$ (the minus is to cancel a positive gust) allows the integral in Eq. (8) to be performed readily, giving

$$\phi_s^{(0)}(x,0) = (bw_0/k)e^{-ikx} \tag{14}$$

Because the upwash extends over all $x$, Eq. (14) is the result that would be produced by a sinusoidal gust in the presence of an infinite plate. Introducing $\phi_s^{(0)}$ into Eq. (9) gives

$$\phi_s^{(1)}(x,0) = (1-i)E[k^*(1-M)x]\phi_s^{(0)}(x,0) \qquad (15)$$

where

$$E(x) = \int_0^x (2\pi t)^{-1/2} e^{it} dt$$

The function $E$ is a combination of Fresnel integrals. It should be noted that there are several definitions of Fresnel integrals in present use. The definition above was chosen because it leads to the simplest form for the resulting equations. $E(x)$ as defined here equals $E^*(x^{1/2})$ of Ref. 9. From Eq. (10), the pressure is found to be

$$P^{(1)}(x,0) = -\rho_0 U w_0 [\pi k x(1+M)]^{-1/2} e^{-i\mu(1-M)x - i\pi/4} \qquad (16)$$

Substitution of this expression into Eq. (12) gives an expression for $p^{(2)}$. However, the following integral results:

$$\int_0^\infty e^{-i2\mu\xi}(r+\xi)^{-1}\xi^{-1/2}(2+\xi)^{-1/2}d\xi$$

Although this could not be evaluated exactly, it could be approximated closely. The largest contributions to the integral come from small $\xi$. Thus, $\xi$ in the radical $2+\xi$ can be neglected, and the result for $P_s^{(2)}$ is

$$P_s^{(2)}(x,0) \approx \rho_0 U w_0 [2\pi k(1+M)]^{-1/2} \{1-(1+i) \times E^*[2\mu(2-x)]\} e^{-i\mu(1-M)x - i\pi/4} + P_s^{(1)}(x,0) \qquad (17)$$

This is seen to go to zero at the trailing edge, as it should, because of the Kutta condition. The * on $E^*$ denotes the complex conjugate.

Expressions for the lift can be obtained by integration of Eqs. (16) and (17). The resulting lift expressions, normalized by the factor $2\pi b\rho_0 U w_0/\beta$, for a gust referenced to the airfoil leading edge are

$$L_s^{(1)} = (1-i)M^{1/2}(\pi\mu\beta)^{-1}E^*[2\mu(1-M)] \qquad (18)$$

$$L_s^{(2)} \approx \frac{i}{\beta(\pi\mu)^{3/2}}\left(\frac{M}{1-M}\right)^{1/2}\left\{\left[\left(\frac{2}{1+M}\right)^{1/2} \times E^*[2\mu(1+M)] - \frac{1-i}{2}\right]e^{-i2\mu(1-M)} + \frac{1+i}{2} - E^*(4\mu)\right\} + L_s^{(1)} \qquad (19)$$

Equations (16-19) for the sinusoidal gust case were derived previously by Adamczyk[9] using the Wiener-Hopf technique. This shows the equivalence between the method of Adamczyk and the present somewhat simpler approach, providing a straightforward method for deriving similar solutions to other upwash cases as will be illustrated. The high frequency asymptote of Eq. (18) is

$$L_s^{(1)} \to -i\beta/(\pi k M^{1/2}) \qquad \text{as } \mu \to \infty \qquad (20)$$

This expression also was given by Drischler.[10] It is interesting to note that Eq. (20) behaves as $k^{-1}$, whereas the classical Sears function, which applies to incompressible flow, behaves as $k^{-1/2}$ for large $k$. Thus, compressibility becomes very important for high frequencies.

The lift expressions given by Eqs. (18-20) are plotted in Fig. 1, along with the numerical results of Graham.[2] In Ref. 6, some justification was given for choosing a value for $\mu$ of the order of $\pi/4$ as the dividing line between the low-frequency solution given in Ref. 5 and the present high-frequency solution. The value of $k$ at which $\mu = \pi/4$ is shown by the solid symbols. The first-order lift $L_s^{(1)}$ gives a result accurate to within about 10% for $\mu > \pi/4$. The use of $L_s^{(2)}$ allows one to calculate accurate results for $\mu$ significantly smaller than $\pi/4$.

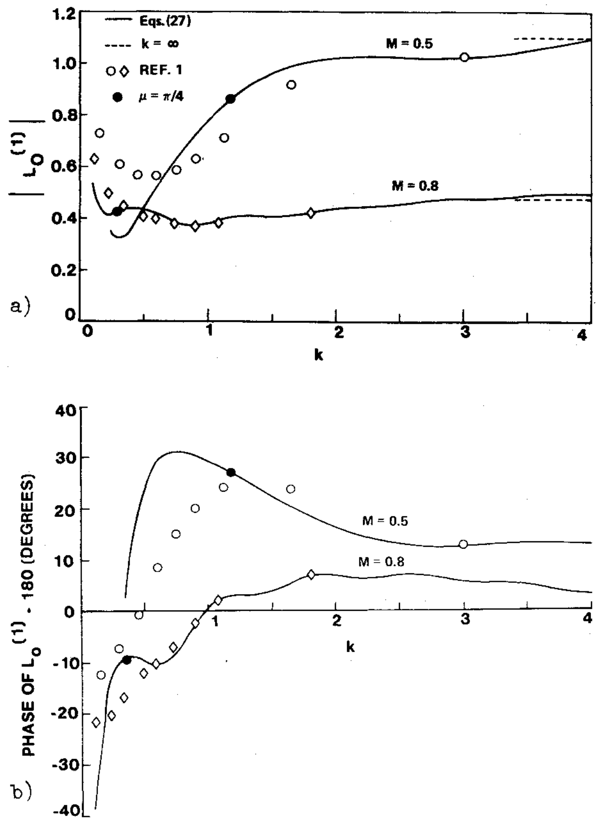


**Fig. 1 Amplitude and phase of the lift due to a sinusoidal gust referenced to the leading edge.**

Since $\phi_s^{(1)}$ satisfies the correct upstream BC only, this is the potential produced by a simi-infinite flat plate in the presence of sinusoidal gust. The solution $\phi_s^{(1)}$ does not satisfy the Kutta condition, but since the solution gives accurate values for large $\mu$ when compared to the numerical results of Graham, one can conclude that the Kutta condition has little effect for the high-frequency Sears gust problem.

An interesting experimental verification of these high-frequency sinusoidal gust response functions is given in Ref. 11, wherein Eq. (19) is used as the airfoil response function in calculating the sound produced by an airfoil in a turbulent flow. The resulting expression shows good agreement with experiment.

The two-dimensional gust problem treated here has wider application than is at first apparent. As Graham[2] has shown, for a gust skewed relative to the airfoil there exists a similarity relation connecting the problem with the two-dimensional gust problem treated here. Also, the case of an infinite span airfoil with finite sweep can be reduced readily to the case of a nonswept airfoil by translating the coordinate system along the airfoil span at a rate such that the mean flow velocity is normal to the span. It was in this generalized form that Eqs. (16-19) for the gust convecting with the freestream were first presented by Adamczyk.[9]

## Solution Procedure for More General Two-Dimensional Upwash

The solution procedure described in the preceding section could be used to obtain a solution for the case of a plunging airfoil (constant upwash) by substituting $w(\xi) = w_0$ into Eq. (8) and proceeding as for the Sears gust case. However, it will be beneficial to modify the solution procedure slightly. For the Sears gust problem, the solution is dominated by leading-edge effects. Thus, correcting first for leading-edge effects, and then correcting for trailing-edge effects, follows the natural ordering of the problem. In contrast, for the case of a plunging airfoil, the leading- and trailing-edge solutions are of the same order. The procedure used for the Sears gust problem would result in solutions that are unnecessarily complex in achieving a given order of accuracy.

The modified solution procedure to be introduced here is to employ the zeroth-order solution to calculate both a leading- and a trailing-edge correction. Thus, Eqs. (8) and (9) are unchanged, but in Eq. (12) $P^{(1)}$ will be replaced by $P^{(0)}$. The sum of the leading- and trailing-edge corrections then will be called a first-order solution $P^{(1)}$. The same general iteration scheme as before now can be applied to obtain higher-order corrections, except that now each higher-order correction will consist of both a leading-edge and a trailing-edge solution calculated from the solution one order lower.

It is interesting to note that this modified solution procedure would lead to the exact same relations for the Sears gust problem as given by Eqs. (14-19). For the first-order correction, the trailing-edge contribution would be zero, since $P_s^{(0)}$, as calculated from Eq. (14), is zero. Thus, the leading-edge solution given by Eq. (15) would be the only contribution. The second-order correction would have only the trailing-edge contribution.

The linear upwash $w(x)=x$ will be treated in the same manner as the constant upwash; i.e., both the leading- and trailing-edge corrections will be calculated simultaneously. However, if $w(x)=x$ is substituted into Eq. (8), the integral is found to diverge, since $w(x)\to\infty$ as $x\to\infty$. Of course, some other function, say $w(x)=0$, could be specified for points off the airfoil, but then the integral cannot be done in closed form. Also, Landahl[8] notes that this discontinuous upwash would not give the correct leading edge singularity to first order.

#### Generalized Gust Result

A way around this difficulty is first to calculate the result for a generalized gust (also known as a Kemp-type gust)

$$w_g(x) = w_0 e^{-i\lambda x} \tag{21}$$

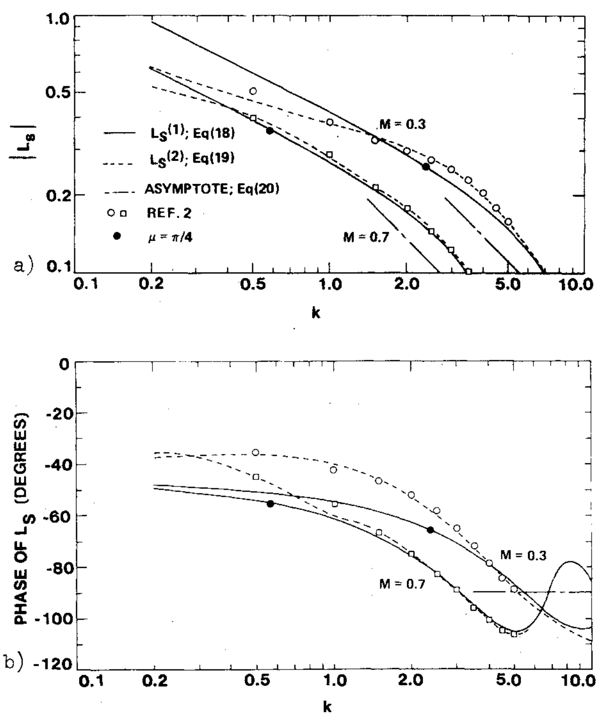


Fig. 2 Amplitude and phase of the lift due to a high frequency plunging motion (constant upwash).

For $\lambda=k$, this is identical to the Sears-type gust considered previously. For $\lambda\neq k$, this represents a sinusoidal gust moving at other than the freestream velocity. Introducing the negative of $w_g$ into Eq. (8) gives

$$\phi_g^{(0)}(x,0) = -(ibw_0/\beta)[\mu^2-(\lambda+M\mu)^2]^{-1/2} e^{-i\lambda x} \tag{22}$$

so that, from Eq. (10),

$$P_g^{(0)}(x,0) = -(i\rho_0 U/b)(k-\lambda)\phi_g^{(0)}(x,0) \tag{23}$$

The leading-edge solution is calculated by use of Eq. (22) in Eq. (9), giving

$$\phi_g^{(0)} + \psi_g^{(1\ell)} = (1+i)E^*[x(\mu(1-M)-\lambda)]\phi_g^{(0)} \tag{24}$$

where the superscript $1\ell$ on $\psi$ denotes a first order leading edge correction. The trailing edge correction is found by use of the modified procedure, substituting Eq. (23) into Eq. (12), which gives

$$p_g^{(1t)} = \{(1+i)E^*[r(\mu(1+M)+\lambda)]-1\}P_g^{(0)} \tag{25}$$

the superscript $1t$ on $p$ denoting that this is a trailing edge correction of first order. The normalized lift expressions calculated from Eqs. (24-25) are

$$L_g^{(0)} + L_g^{(1\ell)} = \frac{1-i}{\pi\lambda}[\mu^2-(\lambda+M\mu)^2]^{-1/2}\left\{k\left(1-\frac{\lambda/\mu}{1-M}\right)^{1/2}\right.$$

$$\left.\times E^*[2\mu(1-M)]-(k-\lambda)e^{-i2\lambda}E^*[2(\mu(1-M)-\lambda)]\right\} \tag{26a}$$

$$L_g^{(1t)} = \frac{i}{\pi}(k(\lambda-1)[\mu^2-(\lambda+M\mu)^2]^{-1/2}\{(1+i)$$

$$\times\left(1+\frac{\lambda/\mu}{1+M}\right)^{1/2} e^{-i2\lambda}E^*[2\mu(1+M)]$$

$$-(1+i)E^*[2(\mu(1+M)+\lambda)]+1-e^{-i2\lambda}\} \tag{26b}$$

The total first order lift is the sum of Eqs. (26a) and (26b).

It should be pointed out that the similarity rules of Graham are limited to the convecting gust problem.[2] Thus, they cannot be used to generalize these results for the parallel gust convecting at other than the freestream velocity to the case of a skewed or three-dimensional gust. However, there is no problem in deriving the results for a skewed gust convecting at other than the stream velocity by direct application of the Schwartzschild solution.

#### Constant and Linear Upwash

The reason for deriving results for the Kemp gust upwash now will become apparent. Although divergent integrals are encountered in attempting to directly calculate results for an upwash $x^n$ for $n\neq 0$, Eq. (20), as was noted by Kemp,[12] contains all of the upwash functions $x^n$ as can be seen by expansion in powers of $\lambda$. Thus, Eqs. (24-26) for potential, pressure, and lift can be expanded in powers of $\lambda$ to obtain results for any upwash $x^n$. (Algebraically, it is simpler to expand the expressions for pressure and integrate to find the lift, rather than directly expanding Eqs. (26) for the lift.) The result for the normalized lift of a plunging airfoil with vertical velocity $w_0 \exp(i\omega t)$ is

$$L_0^{(0)} + L_0^{(1\ell)} = -\frac{\beta}{\pi M}(1+i)\left\{\left[2+\frac{i}{2\mu(1+M)}\right]\right.$$

$$\left.\times E^*[2\mu(1-M)]-i[\pi\mu(1-M)]^{-1/2}e^{-i2\mu(1-M)}\right\} \tag{27a}$$

$$L_0^{(1t)} = -\frac{\beta}{\pi M}\{(1+i)[2+\frac{i}{2\mu(1+M)}]$$

$$\times E^*[2\mu(1+M)] + (1-i)[\pi\mu(1+M)]^{-1/2} e^{-i2\mu(1+M)} - 2\} \quad (27b)$$

with $L^{(1)} = L^{(0)} + L^{(1\ell)} + L^{(1t)}$.

The terms of first order in $\lambda$ obtained from Eqs. (26) are for an upwash $ixw_0$. Multiplying by $-i$ and subtracting out the component due to plunging motion (which arises because $x=0$ is located at the leading edge rather than the midchord) gives, for $L_l^{(1)}$, the normalized lift of an airfoil with an upwash $(x-1)w_0 \exp(i\omega t)$,

$$L_l^{(0)} + L_l^{(1\ell)} = \frac{1+i}{4\pi\mu M\beta}\left\{\left[\frac{1+6M-3M^2}{2\mu\beta^2} + 2i(1-M)\right]\right.$$

$$\left.\times E^*[2\mu(1-M)] - (1+M)[\pi\mu(1-M)]^{-1/2} e^{-i2\mu(1-M)}\right\} \quad (28a)$$

$$L_l^{(1t)} = \frac{(1+i)(1-M)}{4\pi\mu M\beta}\left\{-\left[\frac{1}{2\mu(1+M)} + 2i\right]\right.$$

$$\left.\times E^*[2\mu(1+M)] + [\pi\mu(1+M)]^{-1/2} e^{-i2\mu(1+M)}\right\} \quad (28b)$$

A word of caution in the use of Eqs. (28) should be added. If these two equations are expanded for large $\mu$, it will be found that the leading term of Eq. (28a) exactly cancels that of Eq. (28b). Since the dominant terms cancel, the accuracy that can be expected from Eqs. (28) is significantly reduced, and it may be necessary to carry the iteration process a step further to obtain accurate results. The same statement holds true for the moment for the plunging airfoil which, by the reciprocity relation of Fettis,[13], is equal to the lift for the linear upwash. The airfoil pressure calculated from Eqs. (24) and (25) for the plunging and linear upwash cases is expected to be reasonably accurate, however, since it is only the cancellation of the leading- and trailing-edge effects which reduces the accuracy of the lift for the linear upwash and the moment for the constant upwash cases.

The lift $L_0^{(1)}$ for the high-frequency plunging airfoil given by the sum of Eqs. (27) is compared with the numerical results of Timman et al.[1] The accuracy of $L_0^{(1)}$ is seen to be comparable to the accuracy of $L_s^{(1)}$ for the Sears gust case shown in Fig. 2. If the iteration scheme were to be taken one step further to calculate $L_0^{(2)}$, the result would be expected to achieve significantly better accuracy, comparable to that of $L_s^{(2)}$.

The lift $L^{(1)}$ given by Eqs. (28) for the high-frequency linear upwash is shown in Fig. 3. Comparison is made with the numerical results of Ref. 1 for the moment about the midchord of an airfoil in plunging motion. Because of the cancellation effect discussed previously between the leading- and trailing-edge solutions, the agreement is poorer than that for the cases of plunging motion and Sears gust. (Before plotting the results of Ref. 1, the magnitude was multiplied by $\frac{1}{2}\beta/k$ and 90° was added to the phase for comparison with the present results.)

In taking the limit $k\to\infty$ of the solutions $p^{(1\ell)}$ and $p^{(1t)}$ one notes that they are the same order in $k$. This gives some justification for the simultaneous calculation of the leading and trailing edge solutions as used here. However, because there always remains some fraction of an acoustic wave propagating from the leading edge whose effect is not cancelled at the trailing edge, the Kutta condition is never exactly satisfied. The trailing edge pressure does approach zero for $k\to\infty$ with $w_0$ fixed. The original iteration procedure in which edge effects are calculated alternately would give a solution

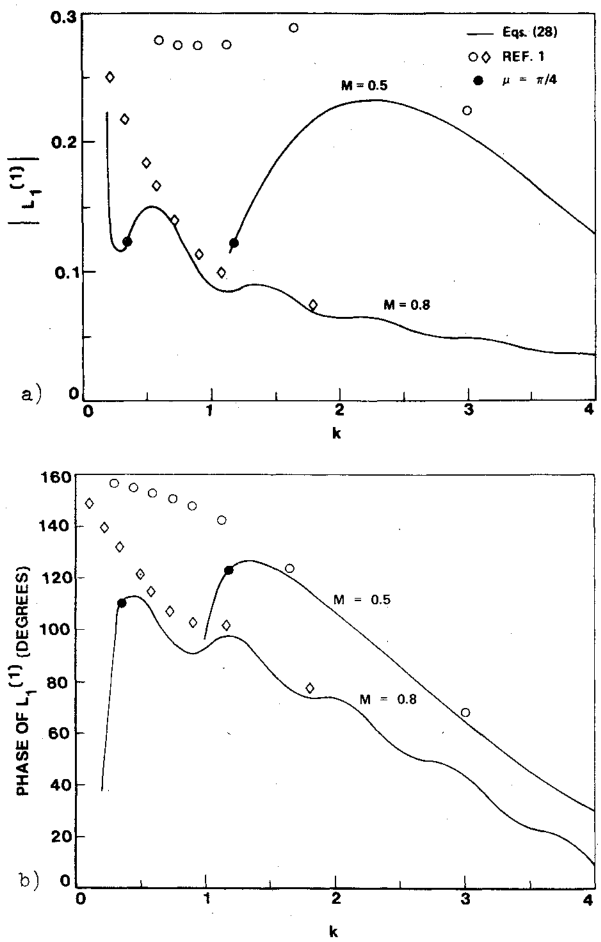


Fig. 3 Amplitude and phase of the lift due to a high frequency linear upwash.

satisfying the Kutta condition when the trailing edge solution is calculated. This original iteration procedure gives somewhat more complex solutions, however, since any given order of solution always contains one extra term. Thus, in the present notation the first order solution, $p^{(0)} + p^{(1\ell)} + p^{(1t)} + p^{(2t)}$, contains the extra term $p^{(2t)}$ when compared with that derived by the simultaneous iteration procedure. The integral in $p^{(2t)}$ presents certain difficulties in its evaluation, also. A calculation of the problems of pitching and plunging was also carried out by Adamczyk[9] (in Appendix IV of the NASA CR only) using the Wiener-Hopf technique together with the original iteration procedure. It appears that the approximation used therein to evaluate the integral for the trailing edge correction resulted in a solution which satisfies the Kutta condition only in the limit $k\to\infty$, as is the case for the solution given here.

## Solution Procedure for a Skewed Gust

The present procedure with slight modifications also can be used to treat the problem of a skewed gust, which has been analyzed previously by Adamczyk,[9] Graham,[14] Filotas,[15] Mugridge[16] and Chu and Widnall.[17] The general relations for $M\neq 0$ will be given and explicit expressions for $M=0$ derived. For a gust of the form

$$w_g = w_0 e^{-i(\lambda x + k_y y)} \quad (29)$$

(where $y$ denotes the spanwise direction) a solution to the wave equation which satisfies the boundary condition on the airfoil can be found by separation of variables to be

$$\phi^{(0)}(x,y,0) = bw_0\beta^{-1}[k_y^2/\beta^2 - \mu^2 + (\mu M+\lambda)^2]^{-1/2} e^{-i(\lambda x + k_y y)} \quad (30)$$

**Table 1 Lift for skewed gust compared with Grahm**

| Wavenumbers | | | Equations (34) | | Graham[2] | |
|---|---|---|---|---|---|---|
| $k_x$ | $k_y$ | $\lvert L^{(1)}\rvert$ | $\lvert L^{(2)}\rvert$ | Phase (deg) | $\lvert L\rvert$ | Phase (deg) |
| 0 | 0.25 | 0.8692 | 0.7307 | 0 | 0.6762 | 0 |
| 0 | 0.5 | 0.5365 | 0.4948 | 0 | 0.4869 | 0 |
| 0 | 1.0 | 0.3038 | 0.2972 | 0 | 0.2970 | 0 |
| 0 | 1.5 | 0.2092 | 0.2078 | 0 | 0.2078 | 0 |
| 0 | 2.0 | 0.1584 | 0.1581 | 0 | 0.1581 | 0 |
| 0 | 2.5 | 0.1271 | 0.1270 | 0 | 0.1270 | 0 |
| 1.0 | 0.25 | 0.4281 | 0.3599 | −37.38 | 0.3706 | −35.74 |
| 1.0 | 0.5 | 0.3588 | 0.3309 | −31.72 | 0.3322 | −30.95 |
| 1.0 | 1.0 | 0.2555 | 0.2499 | −22.50 | 0.2499 | −22.38 |
| 1.0 | 2.0 | 0.1498 | 0.1495 | −13.28 | 0.1494 | −13.25 |

**Table 2 Quarter chord moment for skewed gust compared with Graham**

| Wavenumbers | | | Equations (35) | | Graham[2] | |
|---|---|---|---|---|---|---|
| $k_x$ | $k_y$ | $-\mathfrak{M}_{re}^{(1)}$ | $-\mathfrak{M}_{re}^{(2)}$ | $\mathfrak{M}_{im}$ | $-\mathfrak{M}_{re}$ | $\mathfrak{M}_{im}$ |
| 0 | 0.25 | −0.0357 | 0.0122 | 0 | | 0 |
| 0 | 0.5 | −0.0020 | 0.0147 | 0 | 0.0134 | 0 |
| 0 | 1.0 | +0.0172 | 0.0204 | 0 | 0.0203 | 0 |
| 0 | 1.5 | 0.0209 | 0.0216 | 0 | 0.0217 | 0 |
| 0 | 2.0 | 0.0206 | 0.0208 | 0 | 0.0209 | 0 |
| 0 | 2.5 | 0.0193 | 0.0193 | 0 | 0.0194 | 0 |

In Eq. (4) $\mu^2$ is replaced by $\mu^2 - k_y^2/\beta^2$. Application of the Schwartzschild solution gives the result that the exponential factor in Eq. (9) should be replaced by

$$-i\mu(1-M) \to -(k_y^2/\beta^2-\mu^2)^{1/2} + i\mu M \tag{31}$$

while that in Eq. (12) should be replaced by

$$-i\mu(1+M) \to -(k_y^2/\beta^2-\mu^2)^{1/2} - i\mu M \tag{32}$$

In the simplest case, where the gust drifts with the freestream ($\lambda = k_x$) and $M=0$, the leading and trailing edge solutions are readily found by use of Eqs. (9, 12, and 30-32), to be

$$P^{(1)} = -\rho_0 U w_0 [\pi x(k_y+ik_x)]^{-1/2} e^{-xk_y} \tag{33a}$$

$$P^{(2)} \approx \{1-(x/2)^{1/2}[1-\mathrm{erf}[(2k_y(2-x))^{1/2}]]\}P^{(1)} \tag{33b}$$

$$L^{(1)} = \pi^{-1}[k_y(k_y+ik_x)]^{-1/2}\mathrm{erf}(2^{1/2}k_y^{1/2}) \tag{34a}$$

$$L^{(2)} \approx (\pi k_y)^{-1}[2\pi(k_y+ik_x)]^{-1/2}\{\mathrm{erf}(2k_y^{1/2}) - 2^{1/2}e^{-2k_y}\mathrm{erf}(2^{1/2}k_y^{1/2}) - 1 + e^{-2k_y}\} + L^{(1)} \tag{34b}$$

The moment $\mathfrak{M}$ about the quarter chord, normalized by $4b^2\pi\rho_0 U w_0$ and measured positive counterclockwise, is

$$\mathfrak{M}^{(1)} = (4\pi k_y)^{-1}[k_y(k_y+ik_x)]^{-1/2}[(1-k_y)\mathrm{erf}(2^{1/2}k_y^{1/2}) - (8k_y/\pi)^{1/2}e^{-2k_y}] \tag{35a}$$

$$\mathfrak{M}^{(2)} \approx (4\pi k_y^2)^{-1}[2\pi(k_y+ik_x)]^{-1/2}\{2(2k_y+1)e^{-2k_y} - 2 + (2-k_y)\mathrm{erf}(2k_y^{1/2}) + k_y(1-e^{-2k_y}) - 2^{1/2}(3k_y+1) \times \mathrm{erf}(2^{1/2}k_y^{1/2})e^{-2k_y} - 4(k_y/\pi)^{1/2}e^{-4k_y}\} + \mathfrak{M}^{(1)} \tag{35b}$$

Equations (33-35) also could have been derived from the corresponding results for a two-dimensional compressible Sears type gust by using the similarity rules of Graham.[2] These equations can be expressed in a form, given by Adamczyk,[9] which gives an approximate solution to the general problem of a convecting skewed gust in compressible flow. This more general solution can also be derived in the above manner by taking $M \neq 0$ in Eqs. (30-32). The present incompressible forms are given to illustrate the method and to show the basic form of the solutions in the elliptic regime. Also, they are needed to illustrate the extremely good accuracy that can be expected of the approximate solutions in the elliptic regime when compared to numerical solutions. From the exponential attenuation of pressure with $x$ in Eq. (33a), excellent accuracy is not surprising for large $k_y$ since the iteration between the leading and trailing edges rapidly converges. The results given by Eqs. (34-35) are compared to the numerical results of Graham[2] in Tables 1 and 2. Both the leading edge solution alone, as well as the sum of the leading and trailing edge solutions, are presented to illustrate the rapid convergence. It will be noted that, for $k_y > 1$, the accuracy is of the order of a few parts in 10,[3] and reasonable accuracy can be obtained for $k_y$ as small as 0.25.

Comparable accuracy would be expected for the case $\lambda \neq k_x$. For this case, as for the two-dimensional generalized gust case, both a leading- and a trailing-edge solution must be included in the first-order solution since the zeroth-order pressure calculated from Eq. (30) is not zero at the trailing edge. The results for a gust of the form of Eq. (29) are

$$P^{(0)} + p^{(1l)} = \rho_0 U w_0\{i(\lambda-k_x)(\lambda^2+k_y^2)^{-1/2}e^{-i\lambda x} \times \mathrm{erf}[x^{1/2}(k_y-i\lambda)^{1/2}] - [\pi x(k_y+i\lambda)]^{-1/2}e^{-xk_y}\} \tag{36a}$$

$$p^{(1t)} = -\rho_0 U w_0 i(\lambda-k_x)(\lambda^2+k_y^2)^{-1/2}e^{-i\lambda x} \times \{1-\mathrm{erf}[r^{1/2}(k_y+i\lambda)^{1/2}]\} \tag{36b}$$

As for the preceding cases, these can be integrated to obtain closed-form approximations to the lift and moment. The sum of Eqs. (36) gives a first-order solution, which would be expected to have an accuracy comparable to Eq. (33a). Again, these results can be generalized to include the effects of compressibility by taking $M \neq 0$ in Eqs. (30-32).

## Summary and Conclusions

An iteration scheme of Schwartzschild and Landahl was adapted to derive closed-form approximate solutions for the high-frequency response of flat-plate airfoil in two-dimensional compressible flow and for the response to a skewed gust of large spanwise wavenumber in incompressible flow. The results for the case of a gust convecting with the freestream agree with results previously given by Adamczyk.

The first-order lift expressions for the Sears gust case and the constant upwash case are accurate to within about 10% for $\mu>\pi/4$. The lift expression for the linear upwash (the moment expression for the constant upwash) shows somewhat poorer agreement with numerical solutions, because of the cancellation of leading and trailing edge effects. The lift expression for a skewed gust in incompressible flow convecting with the freestream shows excellent agreement with numerical solutions for $k_y>0.25$. Although not checked against numerical solutions, the pressure distribution for the cases considered is expected to be reasonably accurate for $\mu>\pi/4$, except near the trailing edge for those cases not satisfying the Kutta condition.

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